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Tits–Kantor–Koecher algebras of strongly prime hermitian Jordan pairs ☆

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Abstract

In this paper we describe the Tits–Kantor–Koecher algebras of strongly prime hermitian Jordan pairs in terms of $*$-prime associative pairs and their Martindale pairs of symmetric quotients.

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Introduction

It is well-known that every Jordan pair gives rise to a 3-graded Lie algebra through the Tits–Kantor–Koecher construction. These Lie algebras, called TKK-algebras for short, have already been studied by several authors, such as Neher [16] and Zelmanov [18,19], although the first one has mainly focused on Lie algebras graded by root systems (related with Jordan pairs covered by grids) while the second one has worked with a larger class of graded Lie algebras.

Regularity properties like semiprimeness, nondegeneracy, primeness and simplicity are inherited by TKK-algebras from their associated Jordan pairs (see [4, 1.6] and Section 1 in this paper). This nice behaviour suggests the use of the structure theory of Jordan pairs to obtain 3-graded Lie algebra analogues. In this paper we describe the TKK-algebras of strongly prime hermitian Jordan pairs. The results obtained here will be used in a forthcoming paper to get descriptions of strongly prime 3-graded Lie algebras. Recall that

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for any strongly prime hermitian Jordan pair V there exists a $*$-prime associative pair R with involution such that $0 \neq H(R,*) \lhd V \leqslant H(Q(R),*)$, where $Q(R)$ is the Martindale pair of symmetric quotients of R. Though in general the TKK construction does not preserve containments, we will show that $\mathrm{TKK}(V)$ is sandwiched between $\mathrm{TKK}(H(R,*))$ and $\mathrm{TKK}(H(Q(R),*))$. Moreover, in the spirit of [3], we will prove a converse of the previous assertion.

We begin with a preliminary section where we review the notions of Jordan 3-graded Lie algebra and TKK-algebra, and we state the connection between both structures. The first section is devoted to studying the transfer of regularity conditions between TKK-algebras and their associated Jordan pairs. To do so we start recalling a result from [4] that relates ideals of a Jordan 3-graded Lie algebra and ideals of its associated Jordan pair. Afterwards, in Section 2, we prove the main theorem of the paper, which relates the TKK-algebra of a strongly prime hermitian Jordan pair to the TKK-algebra of the set of symmetric elements of a $*$-prime associative pair with involution R and to the TKK-algebra of the set of symmetric elements of the Martindale pair of quotients of R. Finally, the last section of the paper studies the converse, proving that strong primeness of a TKK-algebra sandwiched between the TKK-algebra of the set of symmetric elements of an associative pair with involution and the TKK-algebra of the set of symmetric elements of its Martindale pair of quotients can be recovered from $*$-primeness of the associative pair.

1. TKK-algebras and 3-graded Lie algebras

1.1. We will deal with Lie algebras and with associative and Jordan systems over an arbitrary ring of scalars Φ. We will also consider involutions on associative systems and, when dealing with associative pairs, involutions will be assumed to be of polarized type. The reader is referred to [1,5,12,14] for basic results, notation and terminology, though we will stress some notions and basic properties. The identities JPx listed in [12] will be quoted with their original numbering without explicit reference to [12].

- Given a Lie algebra L, its product will be denoted by $[x,y]$, for $x,y \in L$. It satisfies $[x,x]=0$ for any $x \in L$ and the Jacoby identity.
- For a Jordan pair $V=(V^+,V^-)$ we will denote the products by $Q_x y$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations denoted by $Q_{x,z}y = \{x,y,z\} = D_{x,y}z$.

1.2. Let $V=(V^+,V^-)$ be a Jordan pair. For $x \in V^+$, $y \in V^-$, we define $\delta(x,y) := (D_{x,y}, -D_{y,x})$, which is a derivation of V (JP12) called *inner derivation*. We denote by $\mathrm{IDer}\, V$ the Φ-module spanned by all inner derivations of V, which is an ideal of the Lie algebra of derivations of V. On the Φ-module $\mathrm{TKK}(V) := V^+ \oplus \mathrm{IDer}\, V \oplus V^-$ we define a Φ-algebra with product $[\,,\,]$ given by $[x^+ \oplus c \oplus x^-, y^+ \oplus d \oplus y^-] := (c^+(y^+) - d^+(x^+)) \oplus ([c,d] + \delta(x^+,y^-) - \delta(y^+,x^-)) \oplus (c^-(y^-) - d^-(x^-))$ where $x^\sigma, y^\sigma \in V^\sigma$ and $c,d \in \mathrm{IDer}\, V$, $\sigma = \pm$. It is shown, for example in [15, XI], that this product has zero square and satisfies the Jacobi identity, so that $\mathrm{TKK}(V)$ is a Lie algebra.

This type of Lie algebras was first considered by Tits [17], Kantor [6–8] and Koecher [9,10], so that $\mathrm{TKK}(V)$ will be said the *Tits–Kantor–Koecher algebra of* V. In general, a TKK-algebra will be a Lie algebra of the form $\mathrm{TKK}(V)$ for some Jordan pair V which will be called its *associated Jordan pair*.

1.3. A 3-*grading* of a Lie algebra L over Φ is a decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$ where each L_i is a submodule of L for $i = 0, \pm 1$, satisfying $[L_i, L_j] \subseteq L_{i+j}$, where $L_{i+j} = 0$ if $i + j \neq 0, \pm 1$. A Lie algebra is called 3-*graded* if it has a 3-grading.

A 3-graded Lie algebra $L = L_1 \oplus L_0 \oplus L_{-1}$ is called *Jordan* 3-*graded* if $[L_1, L_{-1}] = L_0$ and there exists a Jordan pair structure on (L_1, L_{-1}) whose Jordan product is related to the Lie product by $\{x, y, z\} = [[x, y], z]$ for any $x, z \in L_\sigma$, $y \in L_{-\sigma}$, $\sigma = \pm 1$. In this case, $V = (L_1, L_{-1})$ is called the *associated Jordan pair*. The prototype of a Jordan 3-graded Lie algebra is the TKK-algebra of a Jordan pair V where $L_1 = V^+$, $L_{-1} = V^-$ and $L_0 = \mathrm{IDer}\, V$.

Given a Jordan 3-graded Lie algebra L, if $1/2 \in \Phi$ the product on the associated Jordan pair is unique and given by $Q_x y = \frac{1}{2}\{x, y, x\} = \frac{1}{2}[[x, y], x]$. Conversely, given a 3-graded Lie algebra L this formula defines a pair structure on (L_1, L_{-1}) as soon as $1/6 \in \Phi$ (cf. [16, 1.2]).

The relation between general Jordan 3-graded Lie algebras and TKK-algebras is described below.

1.4. Lemma [16, 1.5(6)]. *Let L be a Jordan* 3-*graded Lie algebra with associated pair V. Then* $\mathrm{TKK}(V) \cong L/C_V$, *where* $C_V = \{x \in L_0 \mid [x, L_1] = 0 = [x, L_{-1}]\} = Z(L) \cap L_0$.

We say that a Lie algebra L is *semiprime* if for any nonzero ideal I of L, $[I, I] \neq 0$, and we say that L is *prime* if for any two nonzero ideals I, J of L, $[I, J] \neq 0$.

1.5. Using 1.4 we have that Jordan 3-graded Lie algebras are not far from the ones directly built out of Jordan pairs by the TKK construction. Indeed, as soon as they are centerfree, for example when they are semiprime, they are isomorphic to TKK-algebras of their associated Jordan pairs.

2. Regularity conditions

Ideals of a Jordan 3-graded Lie algebra $L = L_1 \oplus L_0 \oplus L_{-1}$ and ideals of its associated Jordan pair $V = (L_1, L_{-1})$ are related by the following result, which is proven in [4, 1.5] for Lie superalgebras and Jordan superpairs so that it is also true in the non-supersetting:

2.1. Lemma. *Let $L = L_1 \oplus L_0 \oplus L_{-1}$ be a Jordan* 3-*graded Lie algebra with associated Jordan pair $V = (L_1, L_{-1})$ and assume that $1/2 \in \Phi$.*

(a) *We denote by π_σ, $\sigma = \pm 1$, the canonical projections of L onto L_σ. If I is an ideal of L, then $I \cap V := (I \cap L_1, I \cap L_{-1})$ and $\pi(I) := (\pi_{+1}(I), \pi_{-1}(I))$ are ideals of V and $\pi(I)^3 \subseteq I \cap V \subseteq \pi(I)$.*

(b) *Any ideal $U = (U_1, U_{-1})$ of V generates an ideal $\mathcal{I}(U)$ of L, given by $\mathcal{I}(U) = U_1 \oplus ([U_1, L_{-1}] + [L_1, U_{-1}]) \oplus U_{-1}$. If $U = (U_1, U_{-1})$ and $W = (W_1, W_{-1})$*

are two ideals of V *satisfying* $[U_\sigma, W_{-\sigma}] = 0$ *and* $\{U_\sigma, L_{-\sigma}, W_\sigma\} = 0$ *for* $\sigma = \pm 1$*, then* $[\mathcal{I}(U), \mathcal{I}(W)] = 0$*. In particular, if* L *is the TKK-algebra of* V *then* $[\mathcal{I}(U), \mathcal{I}(\mathrm{Ann}_V(U))] = 0$*, where* $\mathrm{Ann}_V(U)$ *is the annihilator of* U *in* V*.*

As a consequence of 2.1 we obtain

2.2. Proposition [4, 1.6]. *Let* V *be a Jordan pair over* Φ *with* $1/2 \in \Phi$*.*

(a) *If* V *is semiprime then so is* $\mathrm{TKK}(V)$*.*
(b) V *is prime if and only if* $\mathrm{TKK}(V)$ *is prime and* V *is semiprime.*
(c) V *is simple if and only if* $\mathrm{TKK}(V)$ *is simple.*

2.3. A basic fact in the proof of the proposition above is that for semiprime Jordan pairs V, nonzero ideals I of $\mathrm{TKK}(V)$ give rise to nonzero ideals $I \cap V$ of V (cf. [4, 2.6]).

2.4. Remark. The converse of 2.2(a) is not clear. On one hand, it seems difficult to prove that if I is an ideal of a Jordan pair V satisfying

$$\{I^\sigma, I^{-\sigma}, I^\sigma\} = 0, \quad \sigma = \pm,$$

then the ideal $\mathcal{I}(I)$ of $\mathrm{TKK}(V)$ is nilpotent. On the other hand, counterexamples to the converse of 2.2(a) are not known.

Recall that, for a Jordan pair V, an element $x \in V^\sigma$ is called an absolute zero divisor if $Q_x = 0$, and that V is nondegenerate if it has no nonzero absolute zero divisors. Similarly, for a Lie algebra L an element $x \in L$ is an *absolute zero divisor* if $(\mathrm{ad}\, x)^2 = 0$, and L is *nondegenerate* if it has no nonzero absolute zero divisors. Following the proof of [11, Proposition 1.5.3] and adapting it to the particular case of absolute zero divisors, we have

2.5. Lemma. *If* L *is a* 2*-torsion free Lie algebra and* $x \in L$ *satisfies* $(\mathrm{ad}\, x)^2 = 0$*, then* $\mathrm{ad}\, x \,\mathrm{ad}\, y \,\mathrm{ad}\, x = 0$ *for any* $y \in L$*.*

2.6. Proposition. *Let* V *be a Jordan pair without* 2*-torsion. Then* V *is nondegenerate if and only if* $\mathrm{TKK}(V)$ *is nondegenerate.*

Proof. Assume that $\mathrm{TKK}(V)$ is nondegenerate and take an absolute zero divisor $x \in V^\sigma$ of V. Then $[[x, \mathrm{TKK}(V)], x] = [[x, V^{-\sigma}], x] = \{x, V^{-\sigma}, x\} = 2Q_x V^{-\sigma} = 0$, so $x = 0$ since $\mathrm{TKK}(V)$ is nondegenerate.

Conversely, assume that V is nondegenerate and take $x \in \mathrm{TKK}(V)$ satisfying $(\mathrm{ad}\, x)^2\, \mathrm{TKK}(V) = 0$. We know that $\mathrm{TKK}(V)$ is a 3-graded Jordan Lie algebra, so $x = x^+ \oplus x_0 \oplus x^-$ for some $x^\sigma \in \mathrm{TKK}(V)_{\sigma 1} = V^\sigma$, $\sigma = \pm$, and $x_0 \in \mathrm{TKK}(V)_0 = \mathrm{IDer}\, V$. Since $(\mathrm{ad}\, x)^2 = 0$, in particular we have that for $\sigma = \pm$, $0 = (\mathrm{ad}\, x)^2 V^\sigma = [[x, V^\sigma], x] = \{x^{-\sigma}, V^\sigma, x^{-\sigma}\} \oplus ([[x_0, V^\sigma], x^{-\sigma}] + [[x^{-\sigma}, V^\sigma], x_0]) \oplus ([[x_0, V^\sigma], x_0] + \{V^\sigma, x^{-\sigma}, x^\sigma\})$, and $0 = \{x^{-\sigma}, V^\sigma, x^{-\sigma}\} = 2Q_{x^{-\sigma}}V^\sigma$ is the only term in $V^{-\sigma}$. Thus $Q_{x^{-\sigma}}V^\sigma = 0$. Since V is nondegenerate, we have that $x^{-\sigma} = 0$ for $\sigma = \pm$, i.e., $x = x_0 \in \mathrm{TKK}(V)_0 = \mathrm{IDer}\, V$.

Using the Jacobi identity, for $v^+ \in V^+$ and $v^- \in V^-$ we have $0 = (\mathrm{ad}\, x)^2[v^+, v^-] = -[[x, [v^+, v^-]], x] = -[[[x, v^+], v^-], x] - [[v^+, [x, v^-]], x] = -2[[x, v^+], [v^-, x]]$, hence

$$\big[[x, v^+], [v^-, x]\big] = 0 \quad \text{for all } v^+ \in V^+,\ v^- \in V^-. \tag{1}$$

Notice that for $v^\sigma \in V^\sigma$, $[x, v^\sigma] = [x_0, v^\sigma] \in V^\sigma$. Therefore, if we take another element $v^{-\sigma} \in V^{-\sigma}$ and use the Jacobi identity, (1) and 2.5, we obtain

$$\begin{aligned}
2Q_{[x,v^\sigma]}v^{-\sigma} &= \big\{[x, v^\sigma], v^{-\sigma}, [x, v^\sigma]\big\} = \big[\big[[x, v^\sigma], v^{-\sigma}\big], [x, v^\sigma]\big] \\
&= \big[\big[[x, v^{-\sigma}], v^\sigma\big], [x, v^\sigma]\big] + \big[\big[x, [v^\sigma, v^{-\sigma}]\big], [x, v^\sigma]\big] \\
&= \big[\big[[x, v^{-\sigma}], [x, v^\sigma]\big], v^\sigma\big] + \big[[x, v^{-\sigma}], \big[v^\sigma, [x, v^\sigma]\big]\big] \\
&\quad + \big[\big[x, [v^\sigma, v^{-\sigma}]\big], [x, v^\sigma]\big] \\
&= \big[\big[x, [v^\sigma, v^{-\sigma}]\big], [x, v^\sigma]\big] = \big[\big[\big[x, [v^\sigma, v^{-\sigma}]\big], x\big], v^\sigma\big] \\
&\quad + \big[x, \big[\big[x, [v^\sigma, v^{-\sigma}]\big], v^\sigma\big]\big] \\
&= \big[x, \big[\big[x, [v^\sigma, v^{-\sigma}]\big], v^\sigma\big]\big] = -\,\mathrm{ad}\, x\, \mathrm{ad}\, v^\sigma\, \mathrm{ad}\, x\big[v^\sigma, v^{-\sigma}\big] = 0.
\end{aligned}$$

Hence $Q_{[x,v^\sigma]}v^{-\sigma} = 0$ for all $v^{-\sigma} \in V^{-\sigma}$ and using nondegeneracy of V we get that $[x, v^\sigma] = 0$ for any $v^\sigma \in V^\sigma$, $\sigma = \pm$, i.e., $[x, V^\sigma] = 0$ for $\sigma = \pm$.

We have proven that $\mathrm{TKK}(V)$ has no nonzero absolute zero divisors, so it is nondegenerate. □

From 2.2(b) and 2.6 we get the following result.

2.7. Corollary. *Let V be a Jordan pair over Φ with $1/2 \in \Phi$. Then V is strongly prime if and only if* $\mathrm{TKK}(V)$ *is strongly prime.*

3. TKK-algebras of strongly prime hermitian Jordan pairs

Unless stated otherwise, from now on all structures (*Lie algebras and associative and Jordan systems*) *are defined over a ring of scalars Φ containing* $1/2$.

3.1. We will study the form of Lie algebras built out of strong prime hermitian Jordan pairs using the TKK construction. In [1, 5.4] D'Amour proves: *If V is an i-special, prime, hereditarily-semiprime Jordan pair* (*in particular if V is an i-special, strongly prime Jordan pair*) *with nonzero hermitian part, then there exists a $*$-prime associative pair R with involution $*$ such that*

$$H(R, *) \lhd V \leqslant H\big(Q(R), *\big) \tag{1}$$

where $Q(R)$ denotes the Martindale pair of symmetric quotients of R.

We will consider the TKK-algebras of $H(R,*)$, V and $H(Q(R),*)$ and see how they are related. Let $V_1 = H(R,*)$, $V_2 = V$, $V_3 = H(Q(R),*)$. With this notation, (1) is rewritten as

$$V_1 \lhd V_2 \leqslant V_3. \tag{1'}$$

Given a Jordan pair V, we say that a Lie algebra L *contains* V *as a subpair* if $V^+, V^- \subseteq L$ and if $V^+ \oplus [V^+, V^-] \oplus V^-$ is a Jordan 3-graded Lie algebra with associated Jordan pair V, which is a subalgebra of L. Notice that a Jordan pair V is always contained as a subpair at least in $\mathrm{TKK}(V)$.

The three Jordan pairs appearing in (1′) are the Jordan pairs associated to the Jordan 3-graded Lie algebras $V_i^+ \oplus [V_i^+, V_i^-] \oplus V_i^-$, for $i = 1, 2, 3$, where Lie products are taken in any Lie algebra containing V_i, $i = 1, 2, 3$, as a subpair (for example, in $A^{(-)}$, where A is the universal associative enveloping algebra of $Q(R)$). We can use 1.4 to describe their TKK-algebras:

$$\mathrm{TKK}(V_i) \cong \big(V_i^+ \oplus \big[V_i^+, V_i^-\big] \oplus V_i^-\big)/C_{V_i} \tag{2}$$

where $C_{V_i} = \{x \in [V_i^+, V_i^-] \mid [x, V_i^+] = 0 = [x, V_i^-]\}$, $i = 1, 2, 3$.

Firstly, we study how the three Lie algebras $V_i^+ \oplus [V_i^+, V_i^-] \oplus V_i^-$, $i = 1, 2, 3$, are related, and secondly, we study their ideals C_{V_i}, $i = 1, 2, 3$. The idea is to find an analogue of (1) for TKK-algebras.

Our results will be stated in a more general setting and will be applied afterwards to the particular case of (1′).

3.2. Lemma. *If I is an ideal of a Jordan pair V, then $I^+ \oplus [I^+, I^-] \oplus I^-$ is an ideal of $\mathcal{I}(I) = I^+ \oplus ([I^+, V^-] + [V^+, I^-]) \oplus I^-$, which is an ideal of $V^+ \oplus [V^+, V^-] \oplus V^-$, where all Lie products are taken in any Lie algebra containing V as a subpair.*

Proof. It is straightforward, using that I is an ideal of V and 2.1(b) applied to $L = V^+ \oplus [V^+, V^-] \oplus V^-$. $\square$

3.3. Since in 3.1(1′) V_1 is an ideal of V_2,

$$V_1^+ \oplus \big[V_1^+, V_1^-\big] \oplus V_1^- \text{ is a graded subideal of } V_2^+ \oplus \big[V_2^+, V_2^-\big] \oplus V_2^- \tag{1}$$

by 3.2. Moreover, V_2 is a subpair of V_3 so

$$V_2^+ \oplus \big[V_2^+, V_2^-\big] \oplus V_2^- \text{ is a graded subalgebra of } V_3^+ \oplus \big[V_3^+, V_3^-\big] \oplus V_3^-. \tag{2}$$

Let us show how the ideals C_{V_i}, $i = 1, 2, 3$, in 3.1(2) are related.

3.4. Lemma. *If U is an ideal of a nondegenerate Jordan pair V contained as a subpair in a Lie algebra L, then*

$$C_U = C_V \cap \left[U^+, U^-\right],$$

where $C_U = \{x \in [U^+, U^-] \mid [x, U^+] = 0 = [x, U^-]\}$ *and* $C_V = \{x \in [V^+, V^-] \mid [x, V^+] = 0 = [x, V^-]\}$ *and* $[\ ,\ ]$ *denotes the Lie product in* L.

Proof. Assume that U is nonzero, since otherwise our claim is immediate. Obviously, $C_U \supseteq C_V \cap [U^+, U^-]$, so take $x \in C_U \subseteq [U^+, U^-]$. We want to show that $[x, V^+] = 0 = [x, V^-]$. Let $v \in V^\sigma$ and consider $[x, v]$. Since $x \in [U^+, U^-]$ and U is an ideal of V, $[x, v] \in U^\sigma$. Then $Q_{U^{-\sigma}}[x, v] \subseteq [x, Q_{U^{-\sigma}} v] \subseteq [x, U^{-\sigma}] = 0$. Since U is nondegenerate, it implies $[x, v] = 0$. $\square$

Under the conditions of 3.1(1′), since V_1 is an ideal of V_2 and V_2 is nondegenerate, $C_{V_1} = C_{V_2} \cap [V_1^+, V_1^-]$ by 3.4, and by 3.1(2) and 3.3(1), up to isomorphism,

$$\mathrm{TKK}(V_1) \lhd_{3\text{-sub}} \mathrm{TKK}(V_2), \tag{1}$$

where, from now on, we will understand that $\lhd_{3\text{-sub}}$ will mean "is a graded subideal of".

We still have to study C_{V_2} and C_{V_3} to find out how $\mathrm{TKK}(V_2)$ and $\mathrm{TKK}(V_3)$ are related. We will use that $H(R, *) \subseteq V_2 = V$ and $V_3 = H(Q(R), *)$.

3.5. A complete construction and description of Martindale systems of quotients can be found in [13]. In this section we will only use Martindale pairs of symmetric quotients $Q(R)$, and particularly the following property about ideal absorption of elements in $Q(R)$:

(i) Let R be a $*$-prime associative pair with involution $*$. Then R is a subpair of $Q(R)$ and for any nonzero element $a \in Q(R)^\sigma$ there exists a nonzero $*$-ideal $I = I_a$ of R such that [13, 3.20] $0 \neq aR^{-\sigma}I^\sigma + I^\sigma R^{-\sigma}a \subseteq R^\sigma$. Notice that if the element $a \in Q(R)^\sigma$ is zero, then any $*$-ideal I of R satisfies $aR^{-\sigma}I^\sigma + I^\sigma R^{-\sigma}a \subseteq R^\sigma$. Moreover, in any case, the ideal I can be replaced by the ideal $(R^+I^-R^+, R^-I^+R^-)$ of R (which is nonzero by semiprimeness) to assure that $R^\sigma I^{-\sigma}a + aI^{-\sigma}R^\sigma \subseteq R^\sigma$, $R^{-\sigma}aI^{-\sigma} + I^{-\sigma}aR^{-\sigma} \subseteq R^{-\sigma}$.

We also point out the following result about the relation between R and $Q(R)$, which is proven in [3, 2.8] in a triple system setting but whose proof also holds for pairs, with the obvious changes:

(ii) Let R be a $*$-prime associative pair with involution $*$. If K is a nonzero $*$-ideal of R, then, for any $0 \neq x \in Q(R)^\sigma$, $xK^{-\sigma}x \neq 0$, $\sigma = \pm$.

3.6. Another result which will be used afterwards and which is proven in [3, 2.1] for associative triple systems but also valid for associative pairs with the obvious changes, is the following: If R is a $*$-prime associative pair with involution $*$ and J is a nonzero $*$-ideal of R, then the set of symmetric elements of J is a nonzero ideal of $H(R,*)$.

The following result is the analogue of 3.5(ii) for sets of symmetric elements.

3.7. Lemma. *Let R be a $*$-prime associative pair with involution $*$. If I is a nonzero ideal of $H(R,*)$, then for any $0 \neq x \in H(Q(R)^\sigma,*)$, $Q_x I^{-\sigma} \neq 0$ and $Q_{I^{-\sigma}} x \neq 0$, $\sigma = \pm$.*

Proof. Suppose that there exists an element $x \in H(Q(R)^\sigma,*)$ for some $\sigma \in \{+,-\}$ such that $Q_x I^{-\sigma} = 0$.

For the nonzero ideal I of $H(R,*)$ by [2, 3.6] there exists a nonzero $*$-ideal A of R such that, for $\sigma = \pm$, $KP(A^\sigma, H(R,*)) \subseteq I^\sigma$.

Since x is an element in $H(Q(R)^\sigma,*) \subseteq Q(R)^\sigma$, by ideal absorption of elements of $Q(R)$ 3.5(i) we know that there exists a nonzero $*$-ideal L of R such that $L^\sigma R^{-\sigma} x \subseteq R^\sigma$ and $L^{-\sigma} x R^{-\sigma} \subseteq R^{-\sigma}$. Since R is $*$-prime, $B = A \cap L$ is a nonzero $*$-ideal of R and we can find a nonzero $*$-ideal C of R such that $C^\sigma \subseteq B^\sigma B^{-\sigma} B^\sigma$ for $\sigma = \pm$.

Now, if $\alpha, \beta \in C^{-\sigma}$, then $\xi := \alpha x \beta \in C^{-\sigma} x C^{-\sigma} \subseteq B^{-\sigma} L^\sigma L^{-\sigma} x C^{-\sigma} \subseteq B^{-\sigma} R^\sigma C^{-\sigma} \subseteq B^{-\sigma}$, hence $\xi + \xi^* \in KP(B^{-\sigma}, H(R,*)) \subseteq KP(A^{-\sigma}, H(R,*))$. Similarly, $\eta := \alpha^* x \alpha \in C^{-\sigma} x C^{-\sigma} \subseteq B^{-\sigma}$ and it is a symmetric element, so

$$\eta = 1/2\big(\eta + \eta^*\big) \in KP\big(B^{-\sigma}, H(R,*)\big) \subseteq KP\big(A^{-\sigma}, H(R,*)\big).$$

Therefore,

$$\begin{aligned} Q_{Q_x\alpha}\beta &= Q_x Q_\alpha Q_x \beta = x\alpha x\beta x\alpha x = x\big(\alpha x\beta + \beta^* x\alpha^*\big)x\alpha x - x\beta^* x\alpha^* x\alpha x \\ &= \big(Q_x\big(\xi + \xi^*\big)\big)\alpha x - x\beta^* Q_x \eta \\ &\in \big(Q_x KP\big(A^{-\sigma}, H(R,*)\big)\big)\alpha x + x\beta^* Q_x KP\big(A^{-\sigma}, H(R,*)\big) = 0, \end{aligned}$$

using that $Q_x KP(A^{-\sigma}, H(R,*)) \subseteq Q_x I^{-\sigma} = 0$.

We have shown $Q_{Q_x C^{-\sigma}} C^{-\sigma} = 0$. Now C is a nonzero $*$-ideal of R so using 3.5(ii) we obtain $Q_x C^{-\sigma} = 0$ which, by 3.5(ii) again, implies $x = 0$.

Finally, since $Q_x I^{-\sigma}$ is nonzero we can take $y \in I^{-\sigma}$ such that $0 \neq Q_x y \in H(Q(R)^\sigma,*)$. Again, $Q_{Q_x y} I^{-\sigma} \neq 0$ and there exists $z \in I^{-\sigma}$ with

$$0 \neq Q_{Q_x y} z = xyxzxyx = \big(Q_x\{y,x,z\}\big)yx - xz(Q_x Q_y x) \in (Q_x Q_I x)Ix + xI(Q_x Q_I x),$$

whence $Q_I x \neq 0$. □

With these ingredients, we can now relate the ideals $C_{H(R,*)}$ and $C_{H(Q(R),*)}$.

3.8. Proposition. *If R is a $*$-prime associative pair with involution $*$, then*

$$C_{H(Q(R),*)} = \big\{x \in \big[H\big(Q(R)^+,*\big), H\big(Q(R)^-,*\big)\big] \mid \big[x, H\big(R^\sigma,*\big)\big] = 0,\ \sigma = \pm\big\}, \quad (1)$$

where $C_{H(Q(R),*)} = \{x \in [H(Q(R)^+, *), H(Q(R)^-, *)] \mid [x, H(Q(R)^\sigma, *)] = 0, \sigma = \pm\}$, *and Lie products are taken in* $A^{(-)}$, *where* A *is the universal associative enveloping algebra of* $Q(R)$.

Proof. It is clear that the first set in (1) is contained in the second one, so take $x \in [H(Q(R)^+, *), H(Q(R)^-, *)]$ such that $[x, H(R^\sigma, *)] = 0$, $\sigma = \pm$, and let us show that this condition is enough to guarantee that $[x, H(Q(R)^\sigma, *)] = 0$, $\sigma = \pm$.

For an arbitrary element $q \in H(Q(R)^\sigma, *)$ there exists a nonzero $*$-ideal $K = (K^+, K^-)$ of R such that $K^{-\sigma} q K^{-\sigma} \subseteq R^{-\sigma}$. Let $H(K, *)$ be the set of symmetric elements of K, which is nonzero by 3.6. Now $Q_{H(K^{-\sigma},*)}[x, q] = [x, Q_{H(K^{-\sigma},*)} q] \subseteq [x, H(R^{-\sigma}, *)] = 0$, which implies $[x, q] = 0$ by 3.7. $\square$

In the following theorem we summarize the main results of this section.

3.9. Theorem. *If* V *is an* i*-special strongly prime Jordan pair with nonzero hermitian part, then there exists a* $*$*-prime associative pair* R *with involution* $*$ *such that, up to isomorphism,*

$$\mathrm{TKK}\big(H(R, *)\big) \lhd_{3\text{-sub}} \mathrm{TKK}(V) \leqslant \mathrm{TKK}\big(H\big(Q(R), *\big)\big),$$

where all the inclusions above are graded.

Proof. Given a Jordan pair V under these hypothesis, it is proven in [1, 5.4] that there exists a $*$-prime associative pair R with involution $*$ such that $H(R, *) \lhd V \leqslant H(Q(R), *)$. We have seen in 3.4(1) that, up to isomorphism, $\mathrm{TKK}(H(R, *)) \lhd_{3\text{-sub}} \mathrm{TKK}(V)$. Moreover, as an immediate consequence of 3.8 we have that if R is a $*$-prime associative pair and V is a Jordan pair satisfying $H(R, *) \subseteq V \subseteq H(Q(R), *)$, then $C_V = C_{H(Q(R),*)} \cap [V^+, V^-]$, and this last equality, 3.3(2) and 3.1(2) imply that, up to isomorphism, $\mathrm{TKK}(V)$ is a graded subalgebra of $\mathrm{TKK}(H(Q(R), *)$, i.e., $\mathrm{TKK}(H(R, *)) \lhd_{3\text{-sub}} \mathrm{TKK}(V) \leqslant \mathrm{TKK}(H(Q(R), *))$. $\square$

4. Strong Primeness of TKK-algebras of hermitian type

4.1. We will prove in this section that the converse of 3.9 is also true. We start with a Jordan pair V for which there exists a $*$-prime associative pair R with involution such that

$$\mathrm{TKK}\big(H(R, *)\big) \lhd_{3\text{-sub}} \mathrm{TKK}(V) \leqslant \mathrm{TKK}\big(H\big(Q(R), *\big)\big), \tag{1}$$

and prove that this is enough to assure that V is strongly prime (equivalently, by 2.7, $\mathrm{TKK}(V)$ is a strongly prime Lie algebra).

In (1) we will understand that all the inclusions are graded, i.e., $\mathrm{TKK}(H(R, *))_\mu \subseteq \mathrm{TKK}(V)_\mu \subseteq \mathrm{TKK}(H(Q(R), *))_\mu$, $\mu = 0, \pm 1$. In particular, $H(R, *)$ is a subpair of V, which is also a subpair of $H(Q(R), *)$.

Notice that if R is an arbitrary $*$-prime associative pair then, by the proof of 3.9, the pair $V = H(R,*)$ is under the hypothesis of (1) and, in particular, one has the graded inclusion $\mathrm{TKK}(H(R,*)) \leqslant \mathrm{TKK}(H(Q(R),*))$.

Our next result is an analogue for TKK-algebras of ideal absorption in Martindale systems of quotients.

4.2. Proposition. *Let $R \neq 0$ be a $*$-prime associative pair with involution, and $Q(R)$ be the Martindale pair of symmetric quotients of R. Then, for any element $x \in \mathrm{TKK}(H(Q(R),*))$, there exists a nonzero ideal M in $\mathrm{TKK}(H(R,*))$ such that $[x, M] \subseteq \mathrm{TKK}(H(R,*))$.*

Proof. First, notice that $H(R,*) \neq 0$ by 3.6, since R is $*$-prime and nonzero. Given $x \in \mathrm{TKK}(H(Q(R),*))$ we know that there exist $n \in \mathbf{N}$, $x^+, x_i^+ \in H(Q(R)^+,*)$ and $x^-, x_i^- \in H(Q(R)^-,*)$, $i = 1,\dots,n$, such that $x = x^+ \oplus \sum_{1\leqslant i\leqslant n}[x_i^+, x_i^-] \oplus x^-$. To simplify the notation, set $x_0^\sigma := x^\sigma$, $\sigma = \pm$.

By ideal absorption 3.5(i), for any x_i^σ, $\sigma = \pm$, $i = 0,\dots,n$, there exists a nonzero $*$-ideal $J_{\sigma i}$ of R that absorbs x_i^σ into R. Defining J as the intersection of all $J_{\sigma i}$, $\sigma = \pm$, $i = 0,\dots,n$, and considering its set of symmetric elements $H(J,*)$ (which is nonzero by $*$-primeness of R and 3.6), we have that $\{H(J^\sigma,*), H(J^{-\sigma},*), H(J^\sigma,*)\}$, $\sigma = \pm$, are nonzero semi-ideals of $H(R,*)$. Therefore, since $H(R,*)$ is strongly prime, there exists a nonzero ideal K of $H(R,*)$ contained in both $\{H(J^\sigma,*), H(J^{-\sigma},*), H(J^\sigma,*)\}$, $\sigma = \pm$.

Now, using the absorption properties of $H(J,*)$ and the Jacobi identity

$$
\begin{aligned}
\big[x_i^\sigma, K^{-\sigma}\big] &\subseteq \big[x_i^\sigma, \big\{H\big(J^{-\sigma},*\big), H\big(J^\sigma,*\big), H\big(J^{-\sigma},*\big)\big\}\big] \\
&\subseteq \big[\big\{x_i^\sigma, H\big(J^{-\sigma},*\big), H\big(J^\sigma,*\big)\big\}, H\big(J^{-\sigma},*\big)\big] \\
&\quad + \big[\big[H\big(J^{-\sigma},*\big), H\big(J^\sigma,*\big)\big], \big[x_i^\sigma, H\big(J^{-\sigma},*\big)\big]\big] \\
&\subseteq \big[H\big(R^\sigma,*\big), H\big(J^{-\sigma},*\big)\big] \\
&\quad + \big[\big[H\big(J^{-\sigma},*\big), H\big(J^\sigma,*\big)\big], \big[x_i^\sigma, H\big(J^{-\sigma},*\big)\big]\big] \\
&\subseteq \big[H\big(R^\sigma,*\big), H\big(J^{-\sigma},*\big)\big] + \big[\big\{H\big(J^{-\sigma},*\big), x_i^\sigma, H\big(J^{-\sigma},*\big)\big\}, H\big(J^\sigma,*\big)\big] \\
&\quad + \big[H\big(J^{-\sigma},*\big), \big\{x_i^\sigma, H\big(J^{-\sigma},*\big), H\big(J^\sigma,*\big)\big\}\big] \\
&\subseteq \big[H\big(R^+,*\big), H\big(J^-,*\big)\big] + \big[H\big(J^+,*\big), H\big(R^-,*\big)\big].
\end{aligned}
$$

Moreover, using the absorption properties of $H(J,*)$ directly we also have that for all $i = 0,\dots,n$, $\sigma = \pm$, $[x_i^\sigma, ([K^+, H(R^-,*)] + [H(R^+,*), K^-])] \subseteq H(R^\sigma,*)$.

Let $\widehat{K}$ be the ideal of $\mathrm{TKK}(H(R,*))$ generated by K (cf. 2.1(b)). For all $i = 1,\dots,n$, $[[x_i^+, x_i^-], \widehat{K}] \subseteq [[x_i^+, \widehat{K}], x_i^-] + [x_i^+, [x_i^-, \widehat{K}]]$ (by the Jacobi identity) $\subseteq H(R^+,*) + [H(R^+,*), x_i^-] + [x_i^+, H(R^-,*)] + H(R^-,*)$.

Using the Jacobi identity and the fact that $\widehat{K}$ is an ideal of $\mathrm{TKK}(H(R,*))$,

$$
\big[\big[x_i^+, x_i^-\big], \big[\widehat{K}, \widehat{K}\big]\big] \subseteq \big[\big[\big[x_i^+, x_i^-\big], \widehat{K}\big], \widehat{K}\big]
$$

$$\begin{aligned}
&\subseteq \widehat{K} + \big[\big[H\big(R^+,*\big), x_i^-\big], \widehat{K}\big] + \big[\big[x_i^+, H\big(R^-,*\big)\big], \widehat{K}\big] \\
&\subseteq \widehat{K} + \big[H\big(R^+,*\big), \big[x_i^-, \widehat{K}\big]\big] + \big[\widehat{K}, x_i^-\big] + \big[x_i^+, \widehat{K}\big] + \big[\big[x_i^+, \widehat{K}\big], H\big(R^-,*\big)\big] \\
&\subseteq \widehat{K} + \mathrm{TKK}\big(H(R,*)\big) \subseteq \mathrm{TKK}\big(H(R,*)\big).
\end{aligned}$$

Notice that $M = [\widehat{K}, \widehat{K}]$ is nonzero since strong primeness of $H(R,*)$, obtained as a consequence of $*$-primeness of R, transfers to $\mathrm{TKK}(H(R,*))$ (2.7). Moreover, M also absorbs the elements x^+ and x^- into $\mathrm{TKK}(H(R,*))$ since $[x^\sigma, M] \subseteq [x^\sigma, \widehat{K}] \subseteq [x^\sigma, K^{-\sigma}] + [x^\sigma, ([K^+, H(R^-,*)] + [H(R^+,*), K^-])] \subseteq \mathrm{TKK}(H(R,*))$. □

4.3. Corollary. *Let R be a $*$-prime associative pair with involution and let x be a nonzero element of* $\mathrm{TKK}(H(Q(R),*))$. *Then $[x, M] \neq 0$ for any nonzero ideal M of* $\mathrm{TKK}(H(R,*))$.

Proof. Let $x = x^+ \oplus x_0 \oplus x^- \in \mathrm{TKK}(H(Q(R),*))$, $x^\sigma \in H(Q(R)^\sigma,*)$, $\sigma = \pm$, $x_0 \in [H(Q(R)^+,*), H(Q(R)^-,*)]$, and let M be a nonzero ideal of $\mathrm{TKK}(H(R),*)$.

Firstly, let us suppose that $x^+ \oplus x^- \neq 0$ (for example, $x^+ \neq 0$). Since M is a nonzero ideal of $\mathrm{TKK}(H(R,*))$ and this last one is strongly prime by [2, 3.7(i)] and 2.7, we can use 2.3 to get $0 \neq (M \cap H(R^+,*), M \cap H(R^-,*)) \triangleleft H(R,*)$. Now, by 3.7, $0 \neq Q_x(M \cap H(R^{-\sigma},*)) = [[x, M \cap H(R^{-\sigma},*)], x]$, and, in particular, $0 \neq [x, M \cap H(R^{-\sigma},*)] \subseteq [x, M]$.

Otherwise, $x = x_0 \in [H(Q(R)^+,*), H(Q(R)^-,*)] = \mathrm{IDer}\, H(Q(R),*)$, so there must exist some $y \in H(Q(R)^\sigma,*)$ with $[x,y] \neq 0$ for some $\sigma \in \{+,-\}$. Now, by 4.2 we can take a nonzero ideal N of $\mathrm{TKK}(H(R),*)$ such that $[y, N] \subseteq \mathrm{TKK}(H(R),*)$. Since $\mathrm{TKK}(H(R),*)$ is strongly prime, $P = [M \cap N, M \cap N]$ is a nonzero ideal of $\mathrm{TKK}(H(R),*)$, so by the first part of this proof $0 \neq [[x,y], P] \subseteq [[x,P], y] + [x, [y,P]] \subseteq [[x,M], y] + [x, [[y,N], M]] \subseteq [[x,M], y] + [x, M]$ hence $[x,M] \neq 0$. □

Now we study the form of absolute zero divisors in a Lie algebra sandwiched between $\mathrm{TKK}(H(R,*))$ and $\mathrm{TKK}(H(Q(R),*))$, for a $*$-prime associative pair R with involution $*$.

4.4. Proposition. *Let R be a $*$-prime associative pair with involution $*$, and let L be a Lie algebra containing* $\mathrm{TKK}(H(R,*))$ *as a subalgebra and which is a subalgebra of* $\mathrm{TKK}(H(Q(R),*))$. *If $x \in L$ is an absolute zero divisor in L, then*

(i) $x = x_0 \in L \cap [H(Q(R)^+,*), H(Q(R)^-,*)]$, *and*
(ii) $[[H(Q(R)^\sigma,*), x], x] = 0$ *for* $\sigma = \pm$.

Proof. Let $x = x^+ \oplus x_0 \oplus x^- \in \mathrm{TKK}(H(Q(R),*))$, $x^\sigma \in H(Q(R)^\sigma,*)$, $\sigma = \pm$, $x_0 \in [H(Q(R)^+,*), H(Q(R)^-,*)]$. The projection of $[[H(R^{-\sigma},*), x], x]$ to $H(Q(R)^\sigma,*)$ is $[[H(R^{-\sigma},*), x^\sigma], x^\sigma]$, hence $Q_{x^\sigma} H(R^{-\sigma},*) = [[H(R^{-\sigma},*), x^\sigma], x^\sigma] = 0$. By 3.7 it implies $x^+ = 0 = x^-$, i.e., $x = x_0$.

Now, if we choose an arbitrary element $a \in H(Q(R)^\sigma,*)$, by 4.2 there exists a nonzero ideal M of $\mathrm{TKK}(H(R,*))$ such that $[M,a] + [M,x] \subseteq \mathrm{TKK}(H(R,*))$. Moreover, we can consider the nonzero ideal $M^4 = [[[M,M],M],M]$ of $\mathrm{TKK}(H(R,*))$

which also satisfies $[a,[x,[a,[x,M^4]]]] + [a,[x,M^4]] \subseteq \mathrm{TKK}(H(R,*))$. Therefore, $\mathrm{ad}([[a,x],x])^2(M^4) = 0$, and by 3.7 it implies $[[a,x],x] = 0$. □

We can now prove the converse of 3.9.

4.5. Theorem. *If R is a $*$-prime associative pair with involution $*$ and V is a Jordan pair such that*

$$\mathrm{TKK}\big(H(R,*)\big) \leqslant \mathrm{TKK}(V) \leqslant \mathrm{TKK}\big(H\big(Q(R),*\big)\big)$$

then $\mathrm{TKK}(V)$ *is strongly prime.* (*The subalgebras above are meant graded, hence* $H(R,*) \leqslant V \leqslant H(Q(R),*)$.)

Proof. Let us first prove that $\mathrm{TKK}(V)$ is nondegenerate. If $x \in \mathrm{TKK}(V)$ is such that $[[\mathrm{TKK}(V),x],x] = 0$, we know that $x = x_0 \in [V^+,V^-]$ by 4.4 taking $L = \mathrm{TKK}(V)$. Therefore, if $y \in V^\sigma$ is an absolute zero divisor in V, by the grading of $\mathrm{TKK}(V)$, also $[[\mathrm{TKK}(V),y],y] = [[V^{-\sigma},y],y] = \{y,V^{-\sigma},y\} = 0$, hence $y \in [V^+,V^-] \cap V^\sigma = 0$. We have shown that V is nondegenerate and by 2.6 this is equivalent to the nondegeneracy of $\mathrm{TKK}(V)$.

To prove that $\mathrm{TKK}(V)$ is prime, take two nonzero ideals of $\mathrm{TKK}(V)$ and let us show that they have nonzero intersection: If I_1 and I_2 are nonzero ideals of $\mathrm{TKK}(V)$, then by 2.3 both $I_1 \cap V$ and $I_2 \cap V$ are nonzero ideals of V. Let y_1 be a nonzero element in $I_1 \cap V^+$ and let y_2 be a nonzero element in $I_2 \cap V^+$. By 4.2, for any of these two elements y_i, $i = 1,2$, which in particular belong to $H(Q(R)^+,*)$, there exists a nonzero ideal H_i of $\mathrm{TKK}(H(R,*))$ such that $0 \neq [y_i,H_i] \subseteq \mathrm{TKK}(H(R,*)) \cap I_i$. Therefore, if we call J_i, $i = 1,2$, the ideals of $\mathrm{TKK}(H(R,*))$ generated by the nonzero subspaces $[y_i,H_i]$ of $\mathrm{TKK}(H(R,*))$, $i = 1,2$, both J_1 and J_2 are nonzero ideals which are contained, respectively, in I_1 and I_2. Now $*$-primeness of R implies that $\mathrm{TKK}(H(R,*))$ is strongly prime by [2, 3.7(i)] and 2.7, hence $0 \neq J_1 \cap J_2 \subseteq I_1 \cap I_2$. □

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